Segment Trees

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Swiss Olympiad in Informatics

Arrays

List of n elements $[x_0, \ldots, x_{n-1}]$ with operations.

Updates

$$[x_0, \ldots, x_i, \ldots, x_{n-1}] \mapsto [x_0, \ldots, \underline{x_i'}, \ldots, x_{n-1}]$$

$$[1, 3, 4, 2, 6, \underbrace{0}_{i=5}, 6, 1, 0, 4] \mapsto [1, 3, 4, 2, 6, \underline{5}, 6, 1, 0, 4]$$

Queries

$$[x_0, \ldots, x_i, \ldots, x_{n-1}] \mapsto x_i$$

$$[1, 3, \underbrace{4}_{i=2}, 2, 6, 5, 6, 1, 0, 4] \mapsto 4$$

Running time: $\mathcal{O}(1)$.

We want to support more general operations.

Operations on Segments

Updates

$$[x_0, \ldots, \underbrace{x_l, \ldots, x_r}_{x \mapsto f(x)}, \ldots, x_{n-1}] \mapsto [x_0, \ldots, \underline{f(x_l), \ldots, f(x_r)}, \ldots, x_{n-1}]$$

$$[1, 3, \underbrace{4, 2, 6, 5}_{x \mapsto x+2}, 6, 1, 0, 4] \mapsto [1, 3, \underline{6, 4, 8, 7}, 6, 1, 0, 4]$$

$$[1, 3, 6, 4, \underbrace{8, 7, 6, 1}_{x \mapsto 3}, 0, 4] \mapsto [1, 3, 6, 4, \underline{3, 3, 3, 3}, 0, 4]$$

Queries

$$[x_0, \ldots, \underbrace{x_l, \ldots, x_r}_{\circ}, \ldots, x_{n-1}] \mapsto x_l \circ \cdots \circ x_r$$

$$[1, 3, \underbrace{6, 4, 3, 3, 3, 3, 0, 4}_{+}] \mapsto 6 + 4 + 3 + 3 + 3 + 0 + 4 = 23$$

$$[1, \underbrace{3, 6, 4, 3, 3}_{\max}, 3, 3, 0, 4] \mapsto \max(\{3, 6, 4, 3, 3\}) = 6$$

Associativity

Queries

$$[x_0, \dots, \underbrace{x_l, \dots, x_r}_{\circ}, \dots, x_{n-1}] \mapsto x_l \circ \dots \circ x_r$$

Why is $x_l \circ x_{l+1} \circ x_{l+2} \circ \dots \circ x_{r-1} \circ x_r$ well-defined?
$\Rightarrow$ We assume that $(x \circ y) \circ z = x \circ y \circ z = x \circ (y \circ z)$.

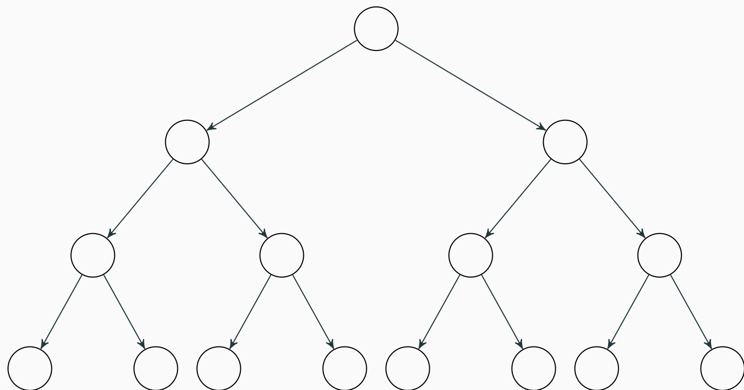
Examples

$+, \cdot, \min, \max, \gcd, \operatorname{lcm}$, AND, OR, XOR, string concatenation, matrix multiplication, function composition

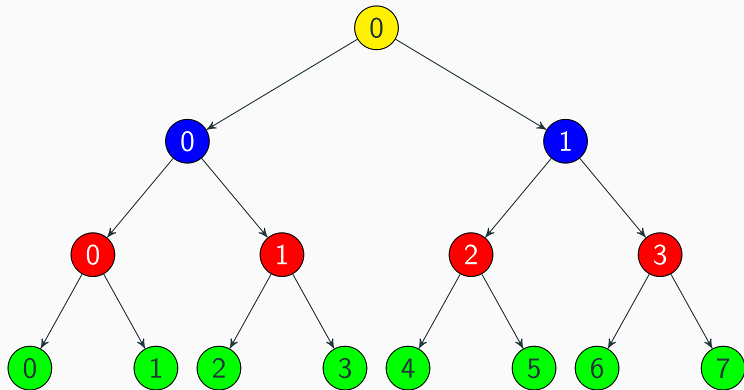
Non-examples

pow, $-$, $/$, average, floating-point addition, NAND, NOR, $==$, ...

Binary Trees



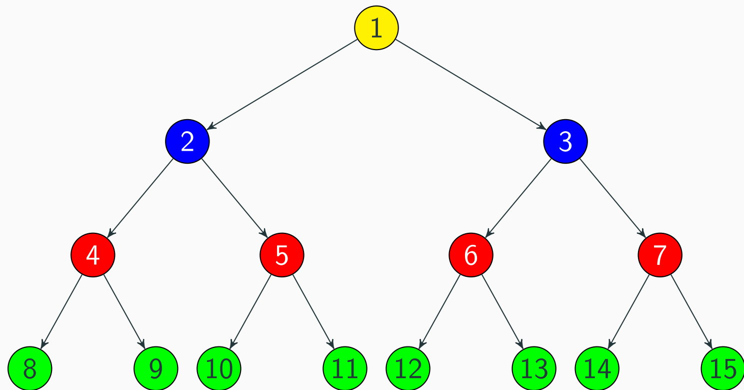
Binary Trees – Layer Structure



For node i:

- children $2 \cdot i$ and $2 \cdot i + 1$ (in next layer)
- parent $\lfloor i/2 \rfloor$

Binary Trees – Layer Structure, Unique Indices



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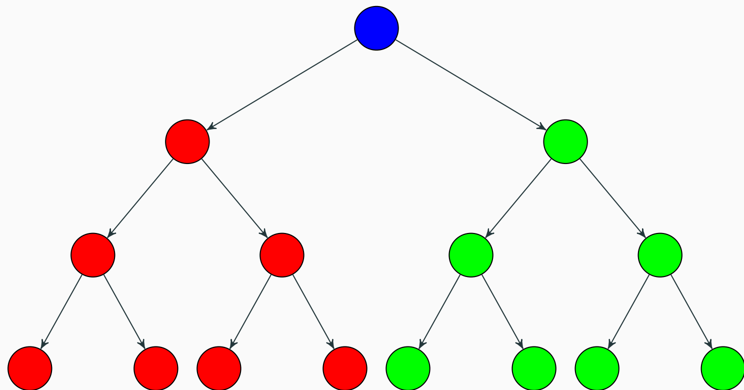
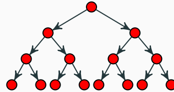
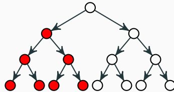
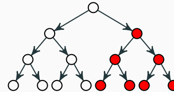
Binary Trees – Recursive Structure

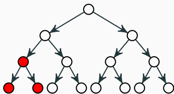


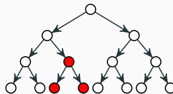
Binary Trees – Subtrees

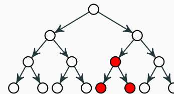


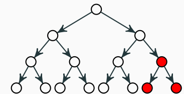


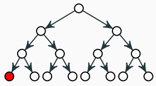


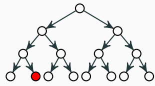


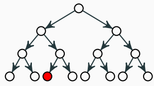


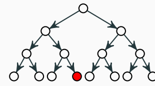


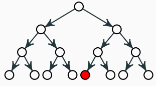


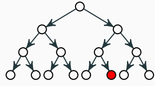


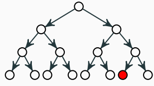


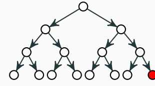






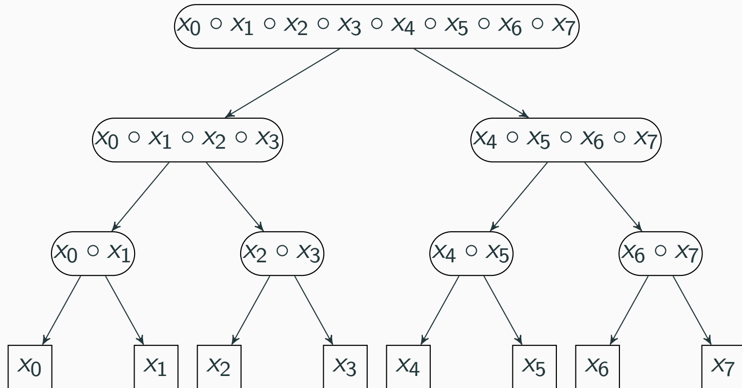




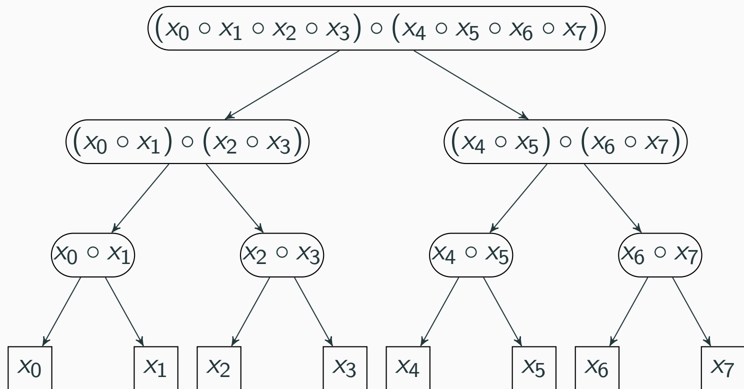


Segment Trees

Idea: Lowest layer in binary tree represents the array, parent summarizes the segment in its subtree.



Building a Segment Tree



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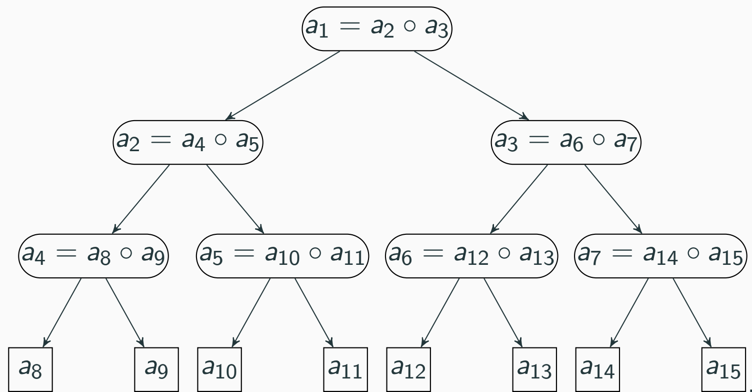
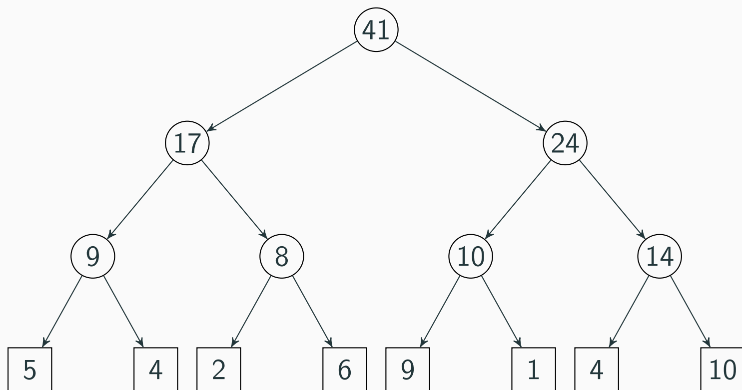
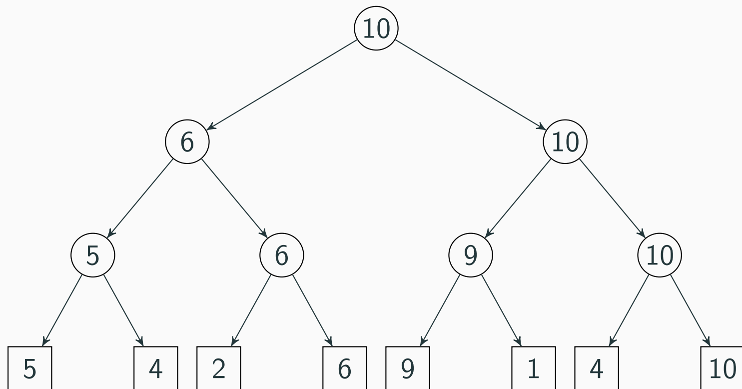


where $[a_8, \ldots, a_{15}] = [x_0, \ldots x_7]$.

Example: Addition

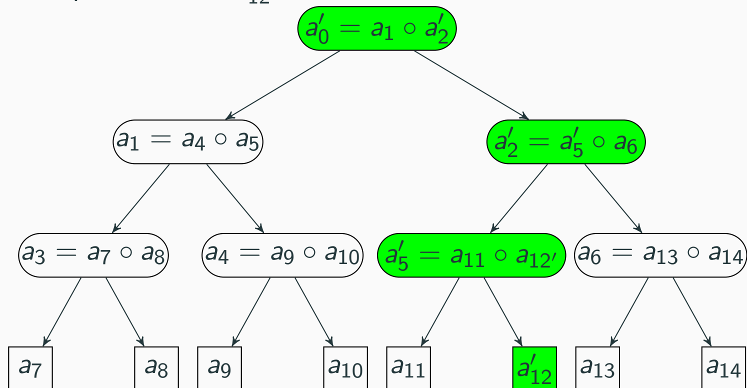


Example: Maximum



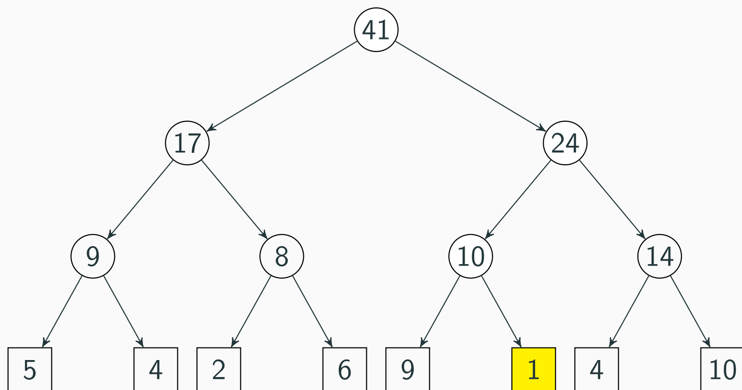
Simple Updates

We update $a_{12} \rightarrow a'_{12}$.



In the segment tree, only one node per layer needs to change.

Example: Addition



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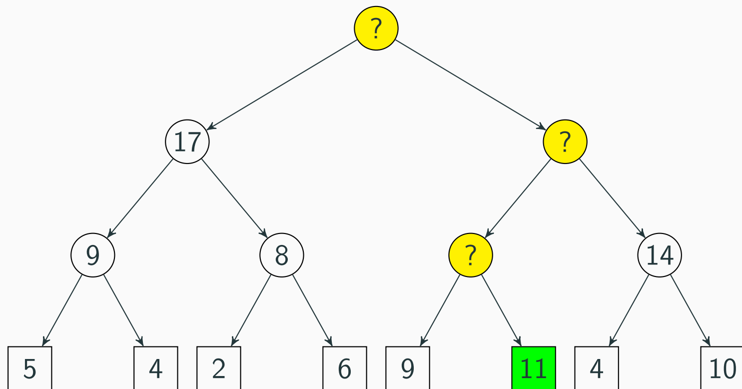

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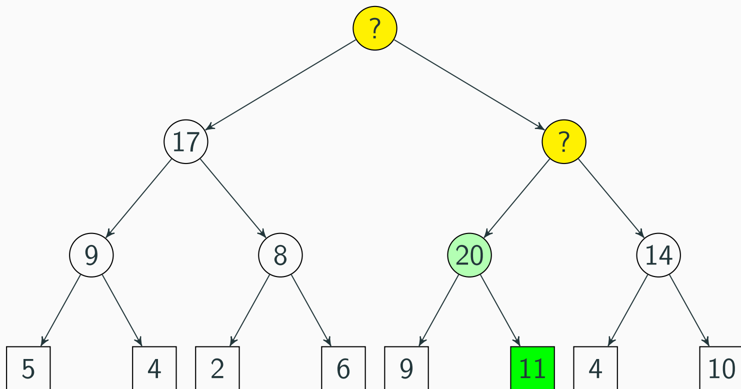

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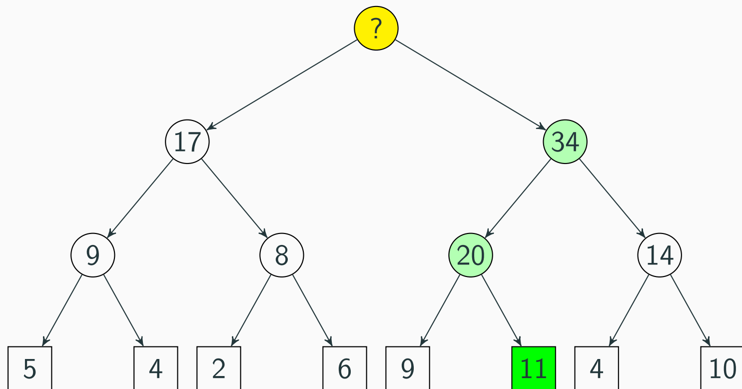

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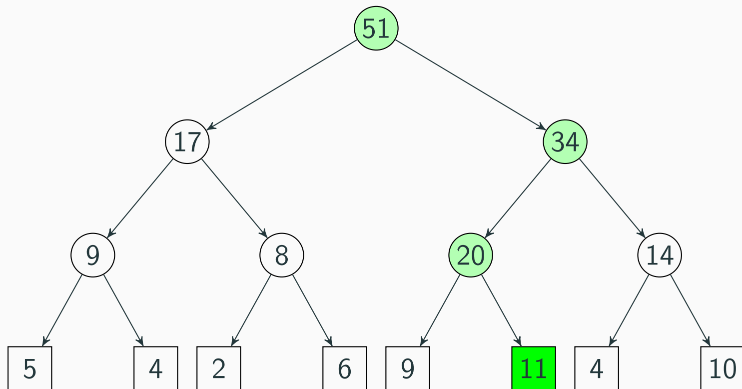
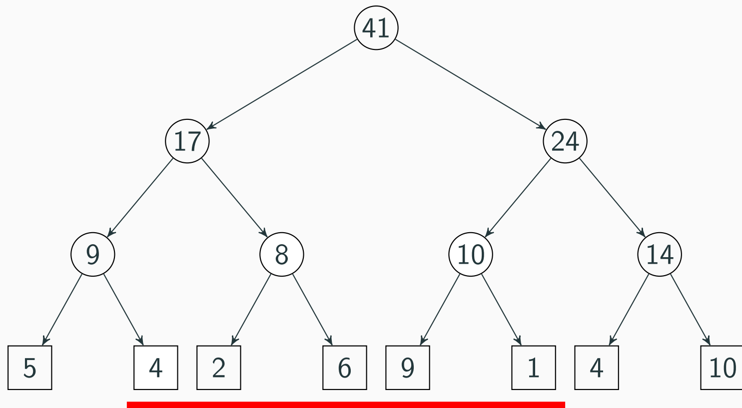


Segment Queries

We have results for a few selected segments.

How to query arbitrary segments?



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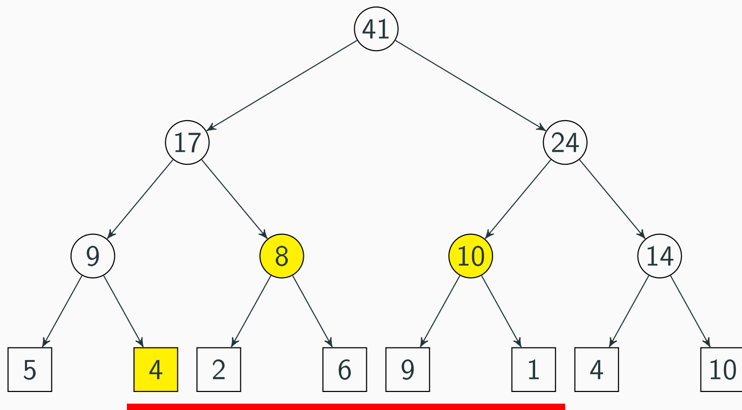
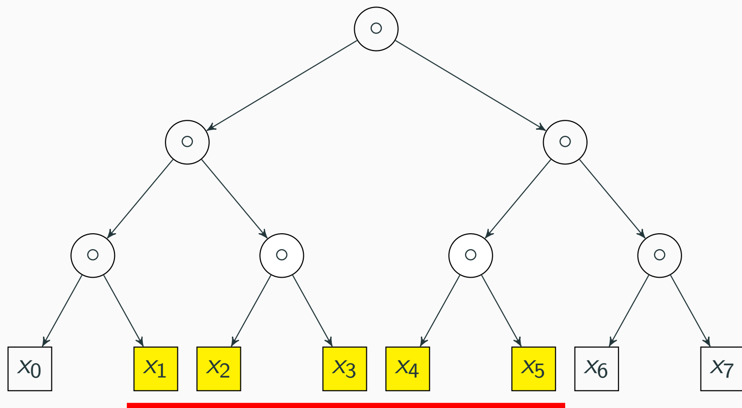


Combine small number of nodes that span the segment together.

$4 + 8 + 10 = 22$.

The Canonical Cover, Example 1

Idea: replace siblings by their parent.



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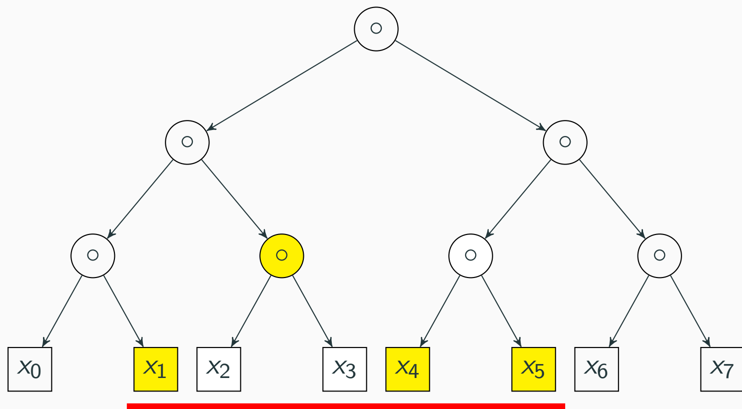

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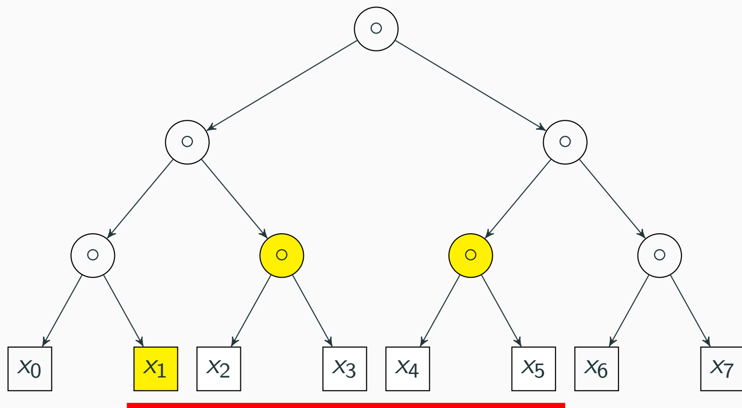
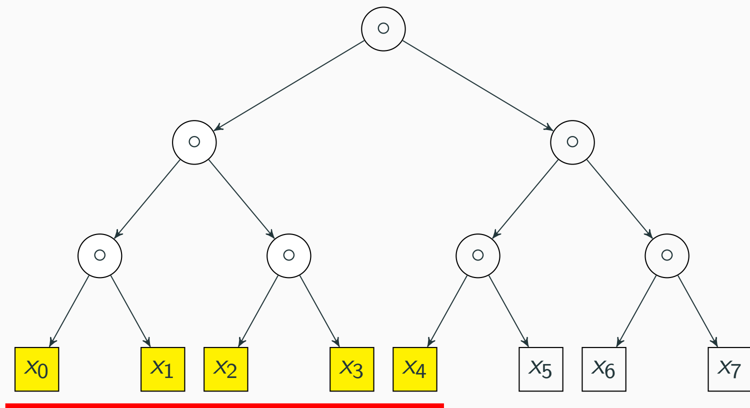


The Canonical Cover, Example 2

Idea: replace siblings by their parent.



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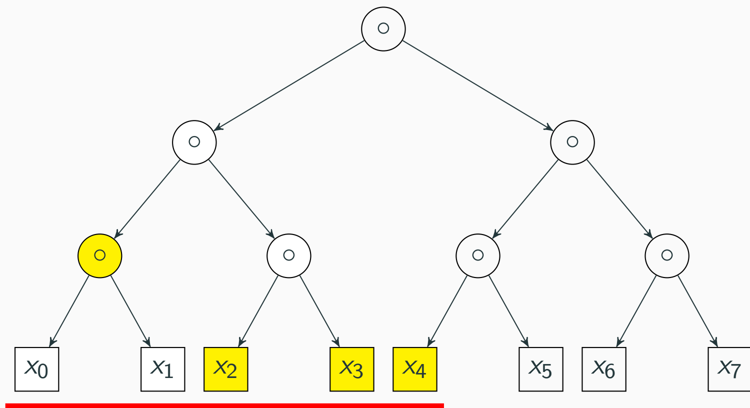

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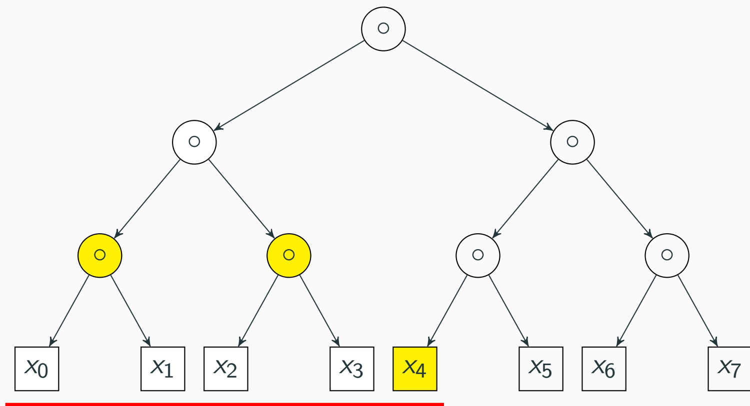

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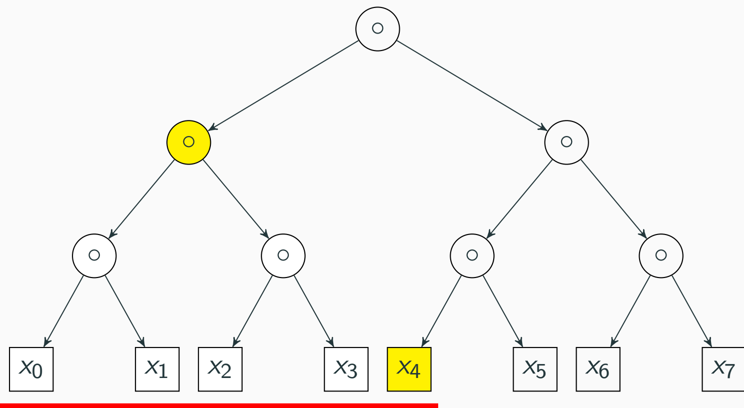
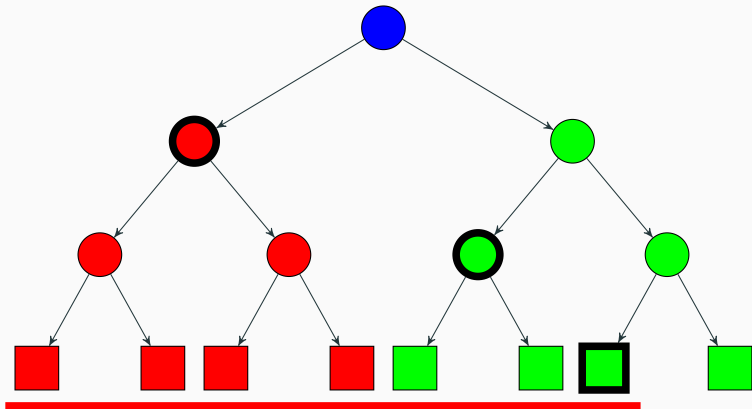


The Canonical Cover is Small

For any segment:

Canonical cover contains at most two nodes per level.

Recursive Computation of Canonical Cover

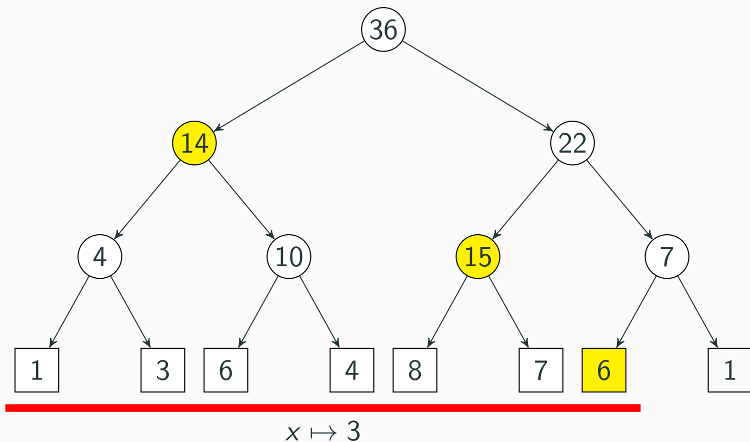


Idea:

- Root in the canonical cover iff the segment spans all leaves.
- Otherwise, take the union of the results for both subtrees.

Segment Updates

Naive idea: Update elements of canonical cover.



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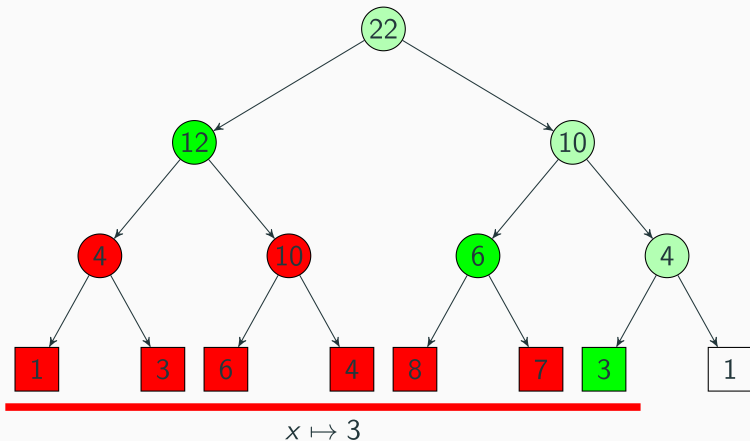
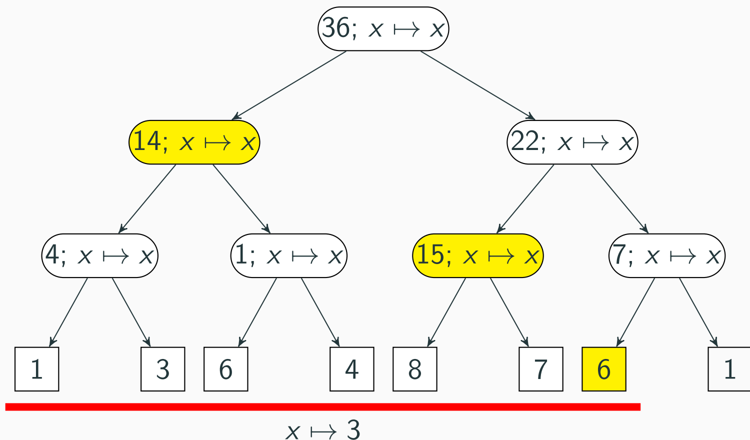


Segment Updates with Lazy Propagation

Idea: Store at each (internal) node an operation that needs to be applied to the child nodes in order to restore the segment tree.



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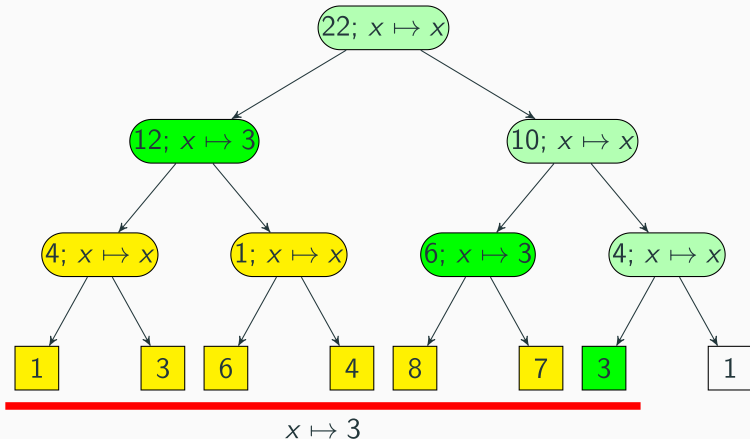
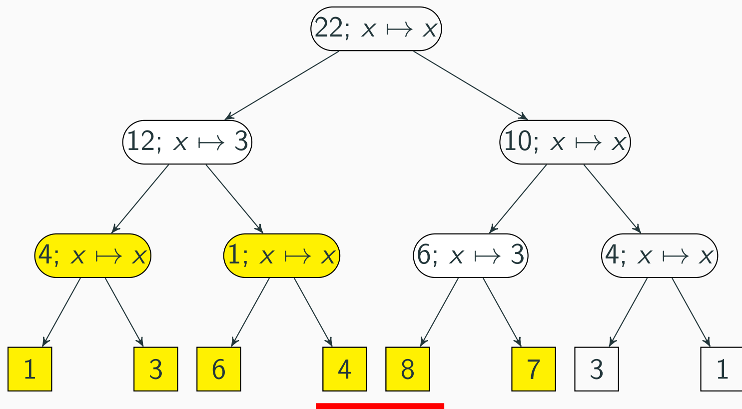


Lazy Propagation during Canonical Cover Computation

Ensure the elements of the computed canonical cover have no pending updates: *Propagate* before recursive calls.



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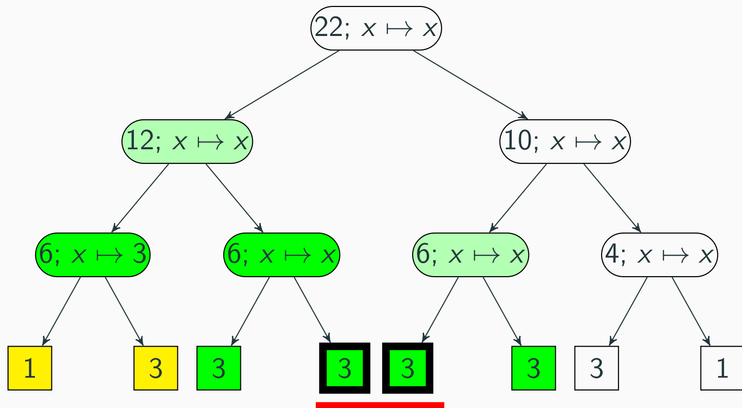


Implementation: Representation

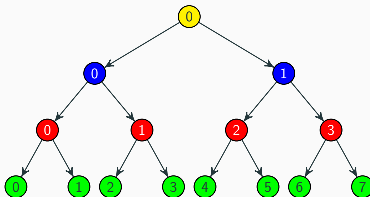
Local node data

```
struct Data{
  Value value;
  Operation lazy;
  int left,right; // left and right borders
  void apply(Operation op);
};
```

(To save space, `left` and `right` can be computed on the fly.)

Implementation: Representation

```
struct SegTree{
  vector<vector<Data>> tree;
};
```



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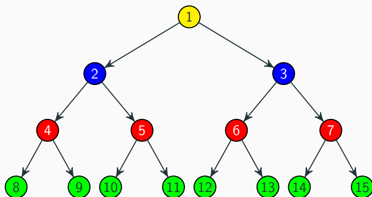

For node i:

- children $2 \cdot i$ and $2 \cdot i + 1$
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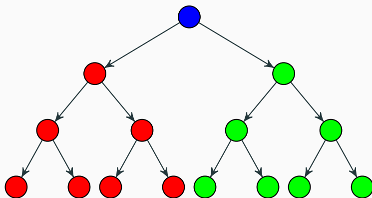
Implementation: Representation

```
struct SegTree{
  Data data;
  SegTree *l,*r; // subtrees
};
```



- Can be initialized lazily (automatic coordinate compression)
- Easily extended to support persistence.

Drawback: Less efficient.

Implementation: Data

We support sum queries, and additive and absolute updates.

```
struct Data{
  int value,left,right;
  pair<bool,int> lazy;
  bool needPropagate()
  { return lazy.first || lazy.second != 0; }
  void apply(pair<bool,int> op){
    if(op.first){ // with reset
      value = op.second*(right-left+1);
      lazy = op;
    }else{
      value += op.second*(right-left+1);
      lazy.second += op.second;
} } };
```

Implementation: Initialization

```
int combine(int a,int b){ return a + b; }
struct SegTree{ // ...
  vector<Data> tree;
  SegTree(vector<int> &data){
    int n=1;
    while(n<2*data.size()) n*=2;
    tree.resize(n);
    for(int i=0;i<n/2;i++){
      tree[n/2+i].value=i<data.size()?data[i]:0;
      tree[n/2+i].left=tree[n/2+i].right=i;
    }
    for(int i=n/2-1;i>=1;i--){
      tree[i].value=combine(tree[2*i].value,
                            tree[2*i+1].value);
      tree[i].left=tree[2*i].left;
      tree[i].right=tree[2*i+1].right;
} } }
```

Implementation: Propagation

```
struct SegTree{
  // ...
  void propagate(int i){
    if(!tree[i].needPropagate()) return;
    for(int k:{0,1}){
      update(2*i+k,tree[i].left,
                   tree[i].right,
                   tree[i].lazy);
    }
    tree[i].lazy={false,0}; // nop
  }
  // ...
}
```

Implementation: Query

```
struct SegTree{ // ...
  int query(int i,int l,int r){
    if(l <= tree[i].left && tree[i].right <= r){
      return tree[i].value;
    }
    if(tree[i].right < l || r < tree[i].left){
      return 0;
    }
    propagate(i);
    return combine(query(2*i,l,r),
                   query(2*i+1,l,r));
  } // ...
}
```

Implementation: Update

```
struct SegTree{// ...
  void update(int i,int l,int r,pair<bool,int> op){
    if(l<=tree[i].left && tree[i].right<=r){
      tree[i].apply(op);
      return;
    }
    if(tree[i].right < l || r < tree[i].left){
      return;
    }
    propagate(i);
    for(int k:{0,1}){
      update(2*i+k,l,r,op);
    }
    tree[i].value=combine(tree[2*i].value,
                          tree[2*i+1].value);
} }
```