Medium Theory Day in Sarnen

Swiss Olympiad in Informatics

13 February 2019

Manual

At the end of the solving phase the tasks will be assigned to the individual group members. (Uniformly at random.)

Evaluation

The presentation should include the following points:

1. Description of the idea and explanation of the algorithm.
2. Proof of why the algorithm is correct.
3. Analysis of asymptotic runtime and memory consumption.

We mainly focus on the runtime and correctness of the algorithm. The quality of the presentation will also be taken into account.

You may show prewritten code, but no other notes. An empty sheet of paper will be available for you during the presentation.

Your presentation should not take longer than 10 minutes.

Technical remarks

It is not stated according to which criteria you should optimize the program. However, the parameter should be clear in all cases (e. g. number n or length of the input string).

You can assume that arithmetic operations with integers can be done in constant time, irrespective of its size. The representation of an integer requires $O(1)$ memory.

Five nights werewolf

Mouse Daniel invented a new game which is called five nights werewolf. Mouse Bibin would technically like to play, but as they are playing during the game jam, he doesn't pay a lot of attention. What he was able to gather is the following:

In the beginning all n mice sit in a row, each of them having a distinct secret number from 1 to n. They are allowed to pick at most one night and in this night move to any place in the row they want (and take their secret number with them). In the end of each night, everyone has to reveal their secret number (now in the changed order) and afterwards, the mice go back to their starting positions.

Task

You are given n and the five sequences of the secret numbers as they were observed after the end of the nights $a_1, \ldots a_n, b_1, \ldots b_n, \ldots e_1, \ldots, e_n$. Mouse Bibin has no idea how the game works exactly, but he wants to find out a possible starting arrangement of the secret numbers. Help him do it. (If there are multiple starting positions that are possible, Bibin is happy with any of them).

Example

$n = 3, 123; 123; 123; 123; 231$ (Anwser: 1 2 3; if all the mice chose to move in the last night, this might have been the starting position)

Bridges

Contrary to popular belief, Sarnen is a small place in the ocean consisting of several islands. After they realized that it is very unpractical to move between the different islands, mouse Benjamin, the mayer of Sarnen, has proposed to build some bridges such that all the islands are connected. He figured out between which of the islands it is even possible the build a bridge. He already received an offer from Stofl's bridge building company. Mouse Stefanie found out that mouse Benjamin will pay them in Sugus, so she also wants to build a bridge. Furthermore,

- mouse Benjamin will select the bridges such that the total building cost is minimized
- nobody knows what goes on in Benjamin's mind - if there are multiple minimizing options it is not clear which one he will take
- she is only allowed to make an offer for a single bridge

Task

You are given n, the number of islands, m, the number of possible bridges and u_i, v_i, c_i, the start and ending point of the bridges resp the number of Sugus that Stofl offered to build this bridge for. Help mouse Stefanie to find the minimal number of Sugus she can certainly get if she offers to build one bridge.

Example

$n = 3, m = 3, 0, 1, 3; 0, 2, 5; 1, 2, 10$ (Answer: 4; Mouse Stefanie can make an offer for the bridge between node 0 and 2 and get 4 Sugus.)

Increasing

Let $D_{n,k}$ be a list of all *increasing* sequences of n integers whose elements are between 1 and k. For example:

$$D_{3,4} = (1,2,3),\ (1,2,4),\ (1,3,4),\ (2,3,4)$$
$$D_{2,5} = (1,2),\ (1,3),\ (1,4),\ (1,5),\ (2,3),\ (2,4),\ (2,5),\ (3,4),\ (3,5),\ (4,5)$$

Within each $D_{n,k}$ the individual sequences have a fixed order: the *lexicographic order*. This order is used in both examples above. Formally, the lexicographic order is defined as follows:

The sequence $a_0, \ldots, a_{n-1}$ is *lexicographically smaller* than the sequence $b_0, \ldots, b_{n-1}$ if there exists an index $j \geq 0$ such that $a_j < b_j$ and for all $i < j$ we have $a_i = b_i$.

For example, $(3,7,12)$ is lexicographically smaller than $(3,8,9)$. Hence, in $D_{3,20}$ the sequence $(3,7,12)$ would appear sooner than $(3,8,9)$. The sequence $(3,7,12)$ is also smaller than $(4,5,6)$.

Task

You are given n, k, and a sequence S which is one of the sequences in $D_{n,k}$. Your task is to find the index of S in $D_{n,k}$. (The first sequence in $D_{n,k}$ has index 1.)

For example, in $D_{3,2}$ the index of $(1,2,4)$ is 2 and the index of $(2,3,4)$ is 4.

Example

$n = 5, k = 8, S = [1,2,3,4,8]$ (Answer: 4; the first four sequences in $D_{5,8}$ are $(1,2,3,4,5)$, $(1,2,3,4,6)$, $(1,2,3,4,7)$, and **(1,2,3,4,8)**.)

Cyclic Shift

The i-th *cyclic shift* of the sequence $A = (a_0, \ldots, a_{n-1})$ is a sequence which is formed by shifting A by i positions to the left, wrapping around. Formally, for $0 \le i < n$ the i-th cyclic shift is defined as $C(A, i) = (a_i, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{i-1})$.

For example, for $A = (3, 4, 5, 6)$ there are 4 different cyclic shifts:

$$\begin{aligned} C(A,0) &= (3,4,5,6) \\ C(A,1) &= (4,5,6,3) \\ C(A,2) &= (5,6,3,4) \\ C(A,3) &= (6,3,4,5) \end{aligned}$$

Note that for some sequences different cyclic shifts may produce the same outcome. For example if $B = (1, 2, 1, 2)$ then $C(B, 0) = C(B, 2) = (1, 2, 1, 2)$ and $C(B, 1) = C(B, 3) = (2, 1, 2, 1)$.

The cyclic shifts of a sequence A can be *ordered lexicographically*. The smallest element in this order is called the *lexicographically smallest cyclic shift*.

Formally, the lexicographic order is defined as follows: The sequence $X = (x_0, \ldots, x_{n-1})$ is *lexicographically smaller* than the sequence $Y = (y_0, \ldots, y_{n-1})$ if there exists an index $j \ge 0$ such that $x_j < y_j$ and for all $i < j$ we have $x_i = y_i$. We will denote this $X \prec Y$.

In the examples above we have $C(A, 0) \prec C(A, 1) \prec C(A, 2) \prec C(A, 3)$ and $C(B, 0) = C(B, 2) \prec C(B, 1) = C(B, 3)$. For the sequence $C = (3, 1, 3, 7)$ the lexicographic order of its cyclic shift is $C(C, 1) \prec C(C, 0) \prec C(C, 2) \prec C(C, 3)$. In particular, note that $C(C, 0) = (3, 1, 3, 7) \prec (3, 7, 3, 1) = C(C, 2)$.

Thus, the lexicographically smallest cyclic shift of A is the shift by 0, for C it is the shift by 1, and for B there are multiple shift amounts that produce the lexicographically smallest cyclic shift: 0 and 2.

Task

There is a hidden sequence A of n integers. Your task is to determine which of its cyclic shifts is the lexicographically smallest one. However, we will not show you the sequence. To obtain the information you need, you will have to ask questions about the sequence. For each question you choose two indices i and j ($0 \le i, j < n$) and call the function CMP(i, j). The return value will be either <, = or >, depending on whether $a_i < a_j$, $a_i = a_j$ or $a_i > a_j$.

Determine for which k, the cyclic shift $C(A, k)$ is the lexicographically smallest one. If there is more than one smallest cyclic shift, find an arbitrary one.