Hard Theory Day in Sarnen

Swiss Olympiad in Informatics

13 February 2019

Manual

At the end of the solving phase the tasks will be assigned to the individual group members. (Uniformly at random.)

Evaluation

The presentation should include the following points:

1. Description of the idea and explanation of the algorithm.
2. Proof of why the algorithm is correct.
3. Analysis of asymptotic runtime and memory consumption.

We mainly focus on the runtime and correctness of the algorithm. The quality of the presentation will also be taken into account.

You may show prewritten code, but no other notes. An empty sheet of paper will be available for you during the presentation.

Your presentation should not take longer than 10 minutes.

Technical remarks

It is not stated according to which criteria you should optimize the program. However, the parameter should be clear in all cases (e. g. number n or length of the input string).

You can assume that arithmetic operations with integers can be done in constant time, irrespective of its size. The representation of an integer requires $O(1)$ memory.

Increasing

Let $D_{n,k}$ be a list of all *increasing* sequences of n integers whose elements are between 1 and k. For example:

$$D_{3,4} = (1,2,3),\ (1,2,4),\ (1,3,4),\ (2,3,4)$$
$$D_{2,5} = (1,2),\ (1,3),\ (1,4),\ (1,5),\ (2,3),\ (2,4),\ (2,5),\ (3,4),\ (3,5),\ (4,5)$$

Within each $D_{n,k}$ the individual sequences have a fixed order: the *lexicographic order*. This order is used in both examples above. Formally, the lexicographic order is defined as follows:

The sequence $a_0, \ldots, a_{n-1}$ is *lexicographically smaller* than the sequence $b_0, \ldots, b_{n-1}$ if there exists an index $j \geq 0$ such that $a_j < b_j$ and for all $i < j$ we have $a_i = b_i$.

For example, $(3,7,12)$ is lexicographically smaller than $(3,8,9)$. Hence, in $D_{3,20}$ the sequence $(3,7,12)$ would appear sooner than $(3,8,9)$. The sequence $(3,7,12)$ is also smaller than $(4,5,6)$.

Task

You are given n, k, and a sequence S which is one of the sequences in $D_{n,k}$. Your task is to find the index of S in $D_{n,k}$. (The first sequence in $D_{n,k}$ has index 1.)

For example, in $D_{3,2}$ the index of $(1,2,4)$ is 2 and the index of $(2,3,4)$ is 4.

Example

$n = 5, k = 8, S = [1,2,3,4,8]$ (Answer: 4; the first four sequences in $D_{5,8}$ are $(1,2,3,4,5)$, $(1,2,3,4,6)$, $(1,2,3,4,7)$, and **(1,2,3,4,8)**.)

Cyclic Shift

The i-th *cyclic shift* of the sequence $A = (a_0, \ldots, a_{n-1})$ is a sequence which is formed by shifting A by i positions to the left, wrapping around. Formally, for $0 \leq i < n$ the i-th cyclic shift is defined as $C(A, i) = (a_i, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{i-1})$.

For example, for $A = (3, 4, 5, 6)$ there are 4 different cyclic shifts:

$$\begin{aligned} C(A, 0) &= (3, 4, 5, 6) \\ C(A, 1) &= (4, 5, 6, 3) \\ C(A, 2) &= (5, 6, 3, 4) \\ C(A, 3) &= (6, 3, 4, 5) \end{aligned}$$

Note that for some sequences different cyclic shifts may produce the same outcome. For example if $B = (1, 2, 1, 2)$ then $C(B, 0) = C(B, 2) = (1, 2, 1, 2)$ and $C(B, 1) = C(B, 3) = (2, 1, 2, 1)$.

The cyclic shifts of a sequence A can be *ordered lexicographically*. The smallest element in this order is called the *lexicographically smallest cyclic shift*.

Formally, the lexicographic order is defined as follows: The sequence $X = (x_0, \ldots, x_{n-1})$ is *lexicographically smaller* than the sequence $Y = (y_0, \ldots, y_{n-1})$ if there exists an index $j \geq 0$ such that $x_j < y_j$ and for all $i < j$ we have $x_i = y_i$. We will denote this $X \prec Y$.

In the examples above we have $C(A, 0) \prec C(A, 1) \prec C(A, 2) \prec C(A, 3)$ and $C(B, 0) = C(B, 2) \prec C(B, 1) = C(B, 3)$. For the sequence $C = (3, 1, 3, 7)$ the lexicographic order of its cyclic shift is $C(C, 1) \prec C(C, 0) \prec C(C, 2) \prec C(C, 3)$. In particular, note that $C(C, 0) = (3, 1, 3, 7) \prec (3, 7, 3, 1) = C(C, 2)$.

Thus, the lexicographically smallest cyclic shift of A is the shift by 0, for C it is the shift by 1, and for B there are multiple shift amounts that produce the lexicographically smallest cyclic shift: 0 and 2.

Task

There is a hidden sequence A of n integers. Your task is to determine which of its cyclic shifts is the lexicographically smallest one. However, we will not show you the sequence. To obtain the information you need, you will have to ask questions about the sequence. For each question you choose two indices i and j ($0 \leq i, j < n$) and call the function CMP(i, j). The return value will be either <, = or >, depending on whether $a_i < a_j$, $a_i = a_j$ or $a_i > a_j$.

Determine for which k, the cyclic shift $C(A, k)$ is the lexicographically smallest one. If there is more than one smallest cyclic shift, find an arbitrary one.

Connectivity

Mouse Lara Gut has a lot of hobbies, she loves to sing and dance, but her real passion is sports, so she could not miss the Olympic Games in Sochi. Arriving at the Olympics, Gut was immediately impressed by the olympics logo: five rings, connected to each other. She realized that she likes connectivity, so she decided to invent a new logo, where everything will be connected.

She decided to start with segments on a line. After she had drawn a few pieces, she wondered how many segments she can select from her set, so that any two selected segments are connected.

Lara believes that two segments are connected, if (and only if) the length of their intersection is greater than zero and less than the length of each segment. That means that the segments must overlap, but neither should be contained in the other. Segments that have exactly one common point are not considered connected.

From the whole set of n segments Lara has drawn, she gives you a list of k segments she likes in particular. For each of these segments she wants you to find the maximal possible set that contains this segment and in which all pairs of segments are connected.

It is guaranteed that no two segments have the same left end, and no two segments have the same right end.

Please help Lara to solve her problem as soon as possible, so that she can go back to winning medals for Switzerland!

Example

$n = 3, s = [(1, 4), (3, 6), (2, 5)], x = [2, 1, 3]$ (Answer: $3, 3, 3$)

Morpher

In this task we are using an alphabet that consists of the first n uppercase letters of the English alphabet. A *morpher* is a function f that is defined on letters of our alphabet. For each letter, f returns a string that consists of at least two letters. An example of a morpher for $n = 3$: $f(\texttt{A}) = \texttt{ABC}$, $f(\texttt{B}) = \texttt{AB}$, $f(\texttt{C}) = \texttt{BC}$.

Once we define a morpher, we can use it to transform strings. The image of a string is obtained by concatenating the images of its letters. For example, when using the morpher defined above, $f(\texttt{BAB}) = f(\texttt{B}) + f(\texttt{A}) + f(\texttt{B}) = \texttt{ABABCAB}$.

As the output uses the same alphabet as the input, it is possible to use the same morpher on a given string several times in a row. For example, $f(f(\texttt{C})) = f(\texttt{BC}) = \texttt{ABBC}$.

You are given a morpher f, a string s, an integer k and an index p. Imagine that we started with the string s and applied the morpher f exactly k times in a row. Return the p-th character (0-based index) of the final string.

If the resulting string has fewer than p characters, output the character "-" instead.

Examples

- $n = 3, k = 1, p = 3, s = \texttt{ABC}, m = [\texttt{ABC}, \texttt{AB}, \texttt{BC}]$ (Answer: `A`).
- $n = 3, k = 1, p = 7, s = \texttt{ABC}, m = [\texttt{ABC}, \texttt{AB}, \texttt{BC}]$ (Answer: `-`).